A new functional analytic approach to robust utility maximization in the dominated case

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I thank the Berlin Mathematical School for full support.

Part of this work was undertaken during a visit to the Hausdorff Research Institute for Mathematics at the University of Bonn within the Trimester Program Stochastic Dynamics in Economics and Finance.

NOVEMBER, 2014

Outline

1. Introduction
   - Utility maximization in continuous time financial markets
   - The convex duality approach
   - Robust problem under “model compactness”
   - Open questions and motivation
2. Robust problem without model compactness
   - A Modular space formulation
   - Our main result
3. Worst-case measure for “linear uncertainty” in complete case
   - Setting and an abstract result
   - Example
4. Conclusions, open problems

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Continuous Time Financial Market

- Filtered probability space $(\Omega, \mathbb{F}, (\mathcal{F})_{t \leq T}, \mathbb{P})$, $\mathbb{P}$ reference law.
- Market consists of d stocks and a risk-less bond, $S = \left(S^i\right)_{0 \leq i \leq d}$.
- S continuous (or loc. bounded) semimartingale.
- The value of portfolio (X_0, π) at time t is $X_t = X_0 + \int_0^t \pi_u dS_u$.
- $\mathcal{M}^e(S) = \left\{\tilde{\mathbb{P}} \sim \mathbb{P} : S \text{ is a } \tilde{\mathbb{P}}\text{-loc. martingale}\right\} \neq \emptyset$.

Admissible wealths starting from x

$$\mathcal{X}(x) = \left\{X \geq 0 : X_t = X_0 + \int_0^t H_u dS_u \text{ with } X_0 \leq x\right\}$$

Utility Functions on $(0, \infty)$

$U : (0, \infty) \to (-\infty, \infty)$ is strictly increasing, strictly concave and continuously differentiable. It satisfies *INADA* if $U'(0+) = \infty$ and $U'(\infty) = 0$. Its asymptotic elasticity is $AE(U) := \limsup_{x \to \infty} \frac{xU'(x)}{U(x)}$.

Utility maximization problems

Standard utility maximization

Agent tries to maximize expected final utility starting from $x > 0$, under the fixed (subjective) model $\mathbb{Q} \cong \mathbb{P}$. Value function is

$$u_{\mathbb{Q}}(x) := \sup_{X \in \mathcal{X}(x)} \mathbb{E}^{\mathbb{Q}}[U(X_T)].$$

Robust utility maximization

Actual probabilistic model (law) possibly unknown (model uncertainty) but there is a set $\mathcal{Q}$ of reasonable possible models.
Pessimistic agent tries to maximize expected final utility of the worst-case model. Value function is

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Duality in financial market models

$$V(y) := \sup_{x>0}[U(x) - xy], y > 0 \text{ conjugate of } U.$$

"Supermartingale densities " w.r.t. (subjective) model $\mathbb{Q}$

$\mathcal{Y}_{\mathbb{Q}}(y) := \{Y \geq 0, YX \text{ is a } \mathbb{Q} - \text{supermartingale } \forall X \in \mathcal{X}(1), Y_0 = y\}$.
Generalizes set of densities wrt. $\mathbb{Q}$ of e.g. risk-neutral measures.

For all $x > 0, X \in \mathcal{X}(x), \mathbb{Q}$,

$$\mathbb{E}^{\mathbb{Q}}[U(X_T)] \leq \inf_{y>0}\left(\inf_{Y\in\mathcal{Y}_{\mathbb{Q}}(y)} \mathbb{E}^{\mathbb{Q}}[V(Y_T)] + xy\right)$$

$$\implies v_{\mathbb{Q}}(y) := \inf_{Y\in\mathcal{Y}_{\mathbb{Q}}(y)} \mathbb{E}^{\mathbb{Q}}[V(Y_T)] \text{ candidate conjugate of } u_{\mathbb{Q}}(x),$$

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Robust case under model compactness

- Dual involves $v(y) = \inf_{\mathbb{Q}\in\mathcal{Q}} \inf_{Y\in\mathcal{Y}_{\mathbb{P}}(y)} \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} V\left(\frac{Y_T}{d\mathbb{Q}/d\mathbb{P}}\right)\right]$.
- Primal requires Minimax: $\sup_{X\in\mathcal{X}(x)} \inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}^{\mathbb{Q}}[U(X_T)] = \inf_{\mathbb{Q}\in\mathcal{Q}} u_{\mathbb{Q}}(x)$.

Conditions on $\mathcal{Q}$ are needed. [SchiedWu05] consider:

1. $\mathcal{Q}$ convex,
2. $\mathbb{P}(A) = 0 \iff \mathbb{Q}(A) = 0 \, \forall \mathbb{Q} \in \mathcal{Q}$, and
3. $\frac{d\mathcal{Q}}{d\mathbb{P}} := \left\{\frac{d\mathbb{Q}}{d\mathbb{P}} : \mathbb{Q} \in \mathcal{Q}\right\}$ closed in $L^0(\mathbb{P})$ (equiv. $\sigma(L^1, L^\infty)-$**compact**).

Theorem ([SchiedWu05] (see also Gundel $\sim$ 03))

Then minimax equality holds and u, v are conjugate. Under additional assumptions (e.g. $AE(U) < 1$), everything is attained:

$$u(x) = u_{\hat{\mathbb{Q}}}(x), \quad \hat{X}_T = (U')^{-1}(\hat{Y}_T/\hat{Z}_T)$$

where $\hat{y} \in \partial u(x)$, $\hat{Y} \in \mathcal{Y}(\hat{y})$ and the pair $\left(\hat{Z} = \frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}, \hat{Y}\right)$ attains the double infimum in the dual problem for such $(x, \hat{y})$.

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Open questions and our motivation

- No general characterization of $\hat{\mathbb{Q}}$.
- There are simple (and reasonable) uncertainty sets, that are not **weakly compact** in $L^1(\mathbb{P})$. e.g.:

$$\mathcal{Q} = \{\mathbb{Q} \ll \mathbb{P} : \mathbb{E}^{\mathbb{Q}}[S_T] \geq A\}, \quad A > 0.$$

  More generally, $\mathcal{Q}$ determined by "moment" or distributional constraints

$$\mathcal{Q} = \bigcap_i \{\mathbb{Q} \ll \mathbb{P} : \mathbb{E}^{\mathbb{Q}}[F_i(S)] \in C_i\}$$

  arise naturally and may fail to be compact.
- **Goal:** Find a framework to study the above problems.
- **Goal:** use general convex duality to describe the worst measure.

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Modular spaces in the robust problem

Assumption

Our U satisfies **INADA**, $U \geq 0$ and $U(\infty) = \infty$.

We know:

$$u(x) = \sup_{X \in \mathcal{X}(x)} \inf_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}^{\mathbb{Q}} [U(X_T)] \leq \inf_{y \geq 0} \left[\inf_{\substack{\mathbb{Q} \in \mathcal{Q}_e \\ Y \in \mathcal{Y}_{\mathbb{P}}(1)}} \mathbb{E}^{\mathbb{P}} \left[\frac{d\mathbb{Q}}{d\mathbb{P}} V \left(\frac{yY_T}{\frac{d\mathbb{Q}}{d\mathbb{P}}} \right) \right] + xy \right],$$

Thus, we care only of $\mathbb{Q} \in \mathcal{Q}$ such that $Z := \frac{d\mathbb{Q}}{d\mathbb{P}}$ belongs to the **Modular** space (more on it later on...):

$$L_I = \left\{ Z \in L^0(\mathbb{P}) \text{ s.t. } \exists \alpha > 0, I(\alpha Z) < \infty \right\}$$

where $I(z) := \inf_{Y \in \mathcal{Y}_{\mathbb{P}}(1)} \mathbb{E}^{\mathbb{P}} \left[|z| V \left(\frac{Y_T(\omega)}{|z|} \right) \right]$.

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Convex Modular

$I : L_I \mapsto [0,\infty]$ is a Convex Modular (and L_I its Modular Space), since:

- $I(0) = 0$
- $I(Z) = I(-Z)$
- For every $Z \in L_I$ there exists $\alpha > 0$ st. $I(\alpha Z) < \infty$
- $[I(\xi Z) = 0$ for every $\xi > 0]$ implies $Z = 0$
- I is convex
- $I(Z) = \sup_{0\leq\xi<1} I(\xi Z)$

We may apply theory of [Musielak] or [Nakano]:

- $|Z|_I^l := \inf\{\alpha > 0 : I(Z/\alpha) \leq 1\}$ and $|Z|_I^a := \inf\left\{\frac{1}{k} + \frac{I(kZ)}{k} : k > 0\right\}$ are equivalent norms

Relation to the robust problem

Suppose $d\mathcal{Q}/d\mathbb{P} \subset L_I$:

- If $Z := d\mathbb{Q}/d\mathbb{P} \in L_I$ then $C(x)|Z|_I \geq u_{\mathbb{Q}}(x) \geq c(x)|Z|_I$
- Notice $v(y) = y \inf_Z I(Z/y)$ and $|Z|_I^a \leq 1 + I(Z)$
- So if L_I is reflexive or I inf-compact, $v(y)$ is attained

$\Rightarrow$ We must explore topology of L_I and related spaces ...
Let us define:

$$E_I = \left\{ Z \in L^0 \text{ s.t. } \forall \alpha > 0, I(\alpha Z) < \infty \right\}$$

$$J(X) = \sup_{Y \in \mathcal{Y}_{\mathbb{P}}(1)} \mathbb{E}[YU^{-1}(X)] \text{ and } L_J, E_J \text{ accordingly}$$

Technical assumption: I, J remain the same when computed using $\{Y \in \mathcal{Y}_{\mathbb{P}}(1) : Y_T > 0 \text{ and } \forall \beta > 0, \mathbb{E}[V(\beta Y)] < \infty\}$ instead of $\mathcal{Y}_{\mathbb{P}}(1)$

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Topology

Under the technical assumption:

Theorem ([B.,Fontbona14])

- *Hölder inequality holds between L_I and L_J*
- *The dual of E_I is isom. isomorphic to L_J*

Most importantly:

Theorem ([B.,Fontbona14])

*If $\mathcal{Y}_{\mathbb{P}}(1)$ is **not** u.i., then E_I and L_I **cannot** be reflexive.*

- In the complete case $\mathcal{Y}_{\mathbb{P}}(1)$ **is** u.i.
- In the incomplete case this happens in pathological cases only.

Main result

Theorem ([B.,Fontbona14])

Under the technical assumption (e.g. $1 \in \mathcal{Y}_{\mathbb{P}}(1)$) and:

- $\mathcal{Q}$ *is convex and* $\mathcal{Q}_e \neq \emptyset$
- $\mathbb{P}(A) = 0 \iff \forall \mathbb{Q} \in \mathcal{Q} : \quad \mathbb{Q}(A) = 0$
- $\frac{d\mathcal{Q}}{d\mathbb{P}} \cap L_I(\mathbb{P})$ *is* $\sigma(L_I, L_J)-$*closed and* $\exists \mathbb{Q} \in \mathcal{Q}_e$ *s.t.* $u_{\mathbb{Q}}(\cdot) < \infty$

If $L_I = E_I$ *(e.g.* $AE(U) < 1$*), then the minimax equality holds, u and v are conjugate and there is an optimal* $X \in \mathcal{X}(x)$.

If further E_I *is reflexive (e.g. market completeness +* $U^{-1} \in \Delta_2$*) then there is a worst* $\hat{\mathbb{Q}} \in \mathcal{Q}$ *and most results in [SchiedWu05] hold also.*

Central arguments:

- $U(\mathcal{X}(x))$ contained in weak*-compact set in L_J.
- Under reflexivity, simply use subsequence principle in E_I.

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Uncertainty set as linear/convex constraints

We consider uncertainty set $\mathcal{Q}$ such that

$$\frac{d\mathcal{Q}}{d\mathbb{P}} = \left\{ \frac{d\mathbb{Q}}{d\mathbb{P}} \in L_I : \Theta\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right) \in C \right\}$$

for $\Theta : L_I(\Omega, \mathbb{P}) \to B$ a linear operator of integral type, taking values in some vector space B (possibly ∞-dim.) and $C \subseteq B$ a convex subset.

More precisely, there is a measurable function $\theta : \Omega \to B$ such that

$$\Theta(Z) = \mathbb{E}^{\mathbb{P}}(Z\theta) \in B$$

This includes moment constraints on " observables" of any dimension; in particular, any restriction (or belief) of distributional type on prices or assets can be described in this way

Uncertainty set as convex constraints: complete case

Minimization problem is embedded into the space $\mathcal{M}_f$ of finite signed measures $\mathbb{M}$ on Ω:

$$\Phi(\mathbb{M}) := \begin{cases} I\left(\frac{d\mathbb{M}}{d\mathbb{P}}\right) = \int \frac{d\mathbb{M}}{d\mathbb{P}} V\left([\frac{d\mathbb{M}}{d\mathbb{P}}]^{-1}\right)\mathbb{P}(d\omega) & \text{if } \mathbb{M} \geq 0 \text{ and } \mathbb{M} \ll \mathbb{P} \\ +\infty & \text{otherwise} \end{cases},$$

adding the constraint $\mathbb{E}^{\mathbb{P}}(\frac{d\mathbb{M}}{d\mathbb{P}}) = 1$. We want:

PC

$$\textit{Minimize}\ \Phi(\mathbb{M}) \quad \text{subject to} \quad \Theta_1(\mathbb{M}) \in C_1 \quad , \mathbb{M} \in \mathcal{M}_f$$

where $\Theta_1(\mathbb{M}) = (\int_\Omega \theta d\mathbb{M}, \int_\Omega 1 d\mathbb{M}) \in B_1 = B \times \mathbb{R}$ and $C_1 = C \times \{1\}$

DC

$$\sup\left\{ \inf_{x \in \bar{B}_1 \cap C_1} \langle g, x \rangle - \int U^{-1}(\langle g, \theta(\cdot)\rangle) d\mathbb{P} : g \in B_1^* \right\}$$

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where $\Theta_1(\mathbb{M}) = (\int_\Omega \theta d\mathbb{M}, \int_\Omega 1 d\mathbb{M}) \in B_1 = B \times \mathbb{R}$ and $C_1 = C \times \{1\}$

DC

$$\sup\left\{ \inf_{x \in \bar{B}_1 \cap C_1} \langle g, x\rangle - \int U^{-1}(\langle g, \theta(\cdot)\rangle) d\mathbb{P} : g \in B_1^* \right\}$$

Uncertainty set as convex constraints: complete case

Minimization problem is embedded into the space $\mathcal{M}_f$ of finite signed measures $\mathbb{M}$ on Ω:

$$\Phi(\mathbb{M}) := \begin{cases} I\left(\frac{d\mathbb{M}}{d\mathbb{P}}\right) = \int \frac{d\mathbb{M}}{d\mathbb{P}} V\left([\frac{d\mathbb{M}}{d\mathbb{P}}]^{-1}\right)\mathbb{P}(d\omega) & \text{if } \mathbb{M} \geq 0 \text{ and } \mathbb{M} \ll \mathbb{P} \\ +\infty & \text{otherwise} \end{cases},$$

adding the constraint $\mathbb{E}^{\mathbb{P}}(\frac{d\mathbb{M}}{d\mathbb{P}}) = 1$. We want:

PC

$$\textit{Minimize}\ \Phi(\mathbb{M}) \quad \text{subject to} \quad \Theta_1(\mathbb{M}) \in C_1 \quad , \mathbb{M} \in \mathcal{M}_f$$

where $\Theta_1(\mathbb{M}) = (\int_\Omega \theta d\mathbb{M}, \int_\Omega 1 d\mathbb{M}) \in B_1 = B \times \mathbb{R}$ and $C_1 = C \times \{1\}$

DC

$$\sup\left\{ \inf_{x \in \bar{B}_1 \cap C_1} \langle g, x\rangle - \int U^{-1}(\langle g, \theta(\cdot)\rangle) d\mathbb{P} : g \in B_1^* \right\}$$

Finding the minimizer

We adapt some results in [Léonard08], since our functions do not fulfil a relevant hypothesis therein...

Theorem ([B.,Fontbona14])

Under above assumptions and ours on U, V:

- *There is dual equality $PC = DC$*
- *If $C_1 \cap \Theta_1(dom(\Phi)) \neq \emptyset$, PC has a unique solution in L_I*
- *If moreover $C_1 \cap icor(\Theta_1(dom(\Phi))) \neq \emptyset$ the solution of PC is given by*

$$\hat{\mathbb{Q}} = \frac{dU^{-1}}{dz}(<\tilde{g}, \theta>)d\mathbb{P}.$$

where $\tilde{g}$ solves DC.

Here, $icor(A) = \{a \in A | \forall x \in aff(A), \exists t > 0 \text{ tq. } a + t(x - a) \in A\}$.

Example

Consider on $\left(\Omega, \mathbb{F}, \{\mathcal{F}_t\}_{t=0}^T, \mathbb{P}\right)$, and for $t \leq T$, the 1d-diffusion

$$dS_t = S_t\{bdt + \sigma dW_t\}, \;\; S_0 = 1$$

Unique risk neutral measure is $d\mathbb{P}^*/d\mathbb{P} = \exp\left\{-\frac{b}{\sigma}W_T - \frac{b^2}{2\sigma^2}T\right\}$.

- We take $U(x) = 2\sqrt{x}$, $x \in (0, \infty)$, thus $L_I = L^2$.
- For $A \geq 0$, consider the uncertainty set

  $$\mathcal{Q} = \{\mathbb{Q} \ll \mathbb{P} : \mathbb{E}^{\mathbb{Q}}(S_T) \geq A\}$$

  which is not closed in L^0 and not bounded in L^2, but is weakly closed in L^2.
- Constraint qualification condition holds by Girsanov Thm.
- We now assume $e^{\sigma^2 T} > A > 1$ for simplicity.

Solution

By Fenchel duality, it follows:

$$\inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}^{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} V\left(yY_T\frac{d\mathbb{P}}{d\mathbb{Q}}\right)\right] = \sup_{\mathbb{R}^2}\left[z_1 + Az_2 - \frac{y}{4}\mathbb{E}^{\mathbb{P}}\left((z_1+S_Tz_2)^2\mathbb{1}_{z_1+S_Tz_2>0}\right)\right]$$

Right-hand side can be solved, and by means of the duality relation between u and v, we get:

$$u(x) = 2\sqrt{x\left(1+\frac{(A-1)^2}{e^{\sigma^2 T}-1}\right)},$$

$$\hat{\mathbb{Q}}(d\omega) = \frac{e^{\sigma^2 T} - A + S_T(A-1)}{e^{\sigma^2 T}-1}\mathbb{P}(d\omega)$$

and

$$\hat{X}_T := x\frac{\left(e^{\sigma^2 T} - A + S_T(A-1)\right)^2}{\left(e^{\sigma^2 T}-1+(A-1)^2\right)\left(e^{\sigma^2 T}-1\right)}.$$

- Functional setting and methodology to solve robust problem in general “markets with uncertainties” is proposed.
- Obtained minimax equality, conjugacy of value functions and existence of optimal wealth... without a worst-case model!
- Currently some classical results can only be recovered in the complete case, and approach is not readily generalizable.
- Worst-case measure can be explicitly (or numerically) computed when uncertainty set is determined by finitely many moment constraints. Expressions hold however in great generality.
- In the non-dominated case one can itroduce similar spaces; nevertheless, the absolute key topological result regarding the indentification of the “dual of L_l” remains ellusive ... and probably would not yield function-like elements!

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